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Almost sure behaviour of the perturbed Brownian motion on the Sierpiński gasket ☆

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Abstract

In this paper we investigate the almost sure ('quenched') asymptotics for a Brownian traveller, moving on the Sierpiński gasket with Poisson-type attracting potential interaction. The quenched behaviour is different from the 'annealed' one (averaged with respect to the random potential).

Keywords: Brownian motion; Sierpiński gasket; Poisson potential

0. Introduction

Let $((X_t)_{t\geqslant 0},\ (P_x)_{x\in\mathscr{G}})$ be the Brownian motion on the infinite Sierpiński gasket $\mathscr{G}$. Suppose that $\mathscr{N}(\omega)=\{x_i\}_{i=1}^{\infty}$ is a Poisson cloud of points falling onto the gasket, defined on some probability space $(\Omega,\mathscr{M},\boldsymbol{P})$, with intensity $v\,\mathrm{d}\mu$ (v is a positive parameter, and μ is the Hausdorff measure on the gasket). We assume that $\mathscr{N}$ is independent of the Brownian motion.

We would like to connect with the cloud $\{x_i\}$ the 'soft-core' potential

$$V(x,\omega)=\sum_{i=1}^{\infty}W(x-x_i), \tag{0.1}$$

with $W:\mathscr{G}\to\mathbb{R}_+$ — a measurable nonnegative function not identically equal to zero, supported inside the ball $B(0,a)$ (a is a given positive number). The situation is more delicate than in the Euclidean space, the Sierpiński gasket not being a translation-invariant set, so that we cannot guarantee that $x-x_i$ belongs to the gasket. To remedy this we define the function W on $\mathbb{R}^2$ and impose the following conditions:

(W1) $W:\mathbb{R}^2\to\mathbb{R}_+$ is measurable,
(W2) $\operatorname{supp} W\subset B(0,a)$ (a is some positive number, fixed from now on),
(W3) W is strictly positive on some $B(0,a_1)$: i.e. $\exists_{0<a_1<a}\exists_{\beta>0}\ \forall_{x\in B(0,a_1)}\ W(x)>\beta$.

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The goal of this paper is to investigate the asymptotic behaviour of the random functional of Feynman-Kac type

$$u^{\omega}(t,z) = E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right]. \tag{0.2}$$

The expected value in (0.2) is taken with respect to the Brownian motion. The functional $u^{\omega}(t,z)$ represents the bounded solution of the following random parabolic equation on $\mathcal{G}$:

$$\partial_t u = \tfrac{1}{2}\Delta u - Vu,$$

$$u_t = 1 \quad \text{for } t = 0.$$

Here Δ denotes the Laplacian on the gasket (Kigami, 1989; Fukushima and Shima, 1992). We will prove the following theorem:

Theorem 0.1. *For* $\boldsymbol{P}$*-almost each* ω *and each* $z \in \mathcal{G}$

$$-C \leqslant \liminf_{t\to\infty} \frac{(\log t)^{2/d_s}}{t} \log E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right]$$

$$\leqslant \limsup_{t\to\infty} \frac{(\log t)^{2/d_s}}{t} \log E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right] \leqslant -D, \tag{0.3}$$

where C, D *are two universal positive constants,* $d_s = 2\log 3/\log 5$ *is the spectral dimension of the gasket.*

According to the physicists' terminology, this is the 'quenched' behaviour, as opposed to the 'annealed' one ('annealed' means 'averaged with respect to the random potential').

Our statement resembles the result of Sznitman (1993), where the precise asymptotic rate for the similar Feynman–Kac functional on $\mathbb{R}^d$ is obtained. We were able to adapt the strategy of the proof of the $\mathbb{R}^d$-case result to the gasket case. The power in which $\log t$ appears is the expected counterpart to that in the Euclidean case, where it appears in the power $2/d$, d being the spectral dimension of $\mathbb{R}^d$.

The correct asymptotics of the expression under study is obtained by considering only those trajectories that spend most of their time in triangles of size approximately equal to $(\log t)^{1/d_f}$, free of Poisson points, lying within distance $[t(\log t)^{1/d_f}]$ from the origin. Here $d_f = \log 3/\log 2$ is the Hausdorff dimension of the gasket.

To get the lower bound we had to overcome the absence of translation invariance of the state space and, consequently, the lack of any Girsanov-type formula, which was used in the Euclidean space. We encountered similar problems before (see Pietruska-Pałuba, 1997). Instead, we discretize the problem and use some hitting time estimates.

The upper bound is obtained via the 'enlarging the obstacles' method (see Sznitman (1990, 1991), and in fractal context Pietruska-Pałuba (1997)). Since the Brownian motion on finitely ramified fractals is point-recurrent, the method can be used in its simpler form. The estimates which allow us to benefit from this technique were obtained in Pietruska-Pałuba (1991).

Recall that for the 'hard-core' potential (killing Poissonian obstacles) we have proved the existence of two positive constants C' and D' such that for each $z \in \mathscr{G}$

$$-C' \leqslant \liminf_{t\to\infty} \frac{\log \boldsymbol{P} \otimes P_z[T > t]}{t^{d_s/(d_s+2)}} \leqslant \limsup_{t\to\infty} \frac{\log \boldsymbol{P} \otimes P_z[T > t]}{t^{d_s/(d_s+2)}} \leqslant -D', \tag{0.4}$$

where T is the entrance time into the obstacles (see Pietruska-Pałuba (1991), Theorems 2 and 6; note that $\boldsymbol{P} \otimes P_z[T > t] = E_z[\mathrm{e}^{-\nu\mu(X_{[0,t]})}]$).

Analogous asymptotics can be proven for the soft potential case, its proof almost unchanged. Precisely, we would get

$$-C' \leqslant \liminf_{t\to\infty} \frac{\log \boldsymbol{E}\left(E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right]\right)}{t^{d_s/(d_2+2)}}$$

$$\leqslant \limsup_{t\to\infty} \frac{\log \boldsymbol{E}\left(E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right]\right)}{t^{d_s/(d_2+2)}} \leqslant -D'.$$

Observe that the almost sure behaviour differs from the averaged behaviour, which is typical in such contexts. To learn more about the Euclidean-space case with Poissonian potential see Sznitman (1993b).

To reduce the volume of the present paper we omit proofs that are basically the same as in the Euclidean space. We refer the reader to Sznitman (1993a).

1. Preliminaries

In this section we set the notation and recall some properties of the Brownian motion on the Sierpiński gasket.

Let $\psi_1, \psi_2, \psi_3 : \mathbb{R}^2 \to \mathbb{R}^2$ be three simulitudes of the plane given by

$$\psi_1(x) = \frac{x}{2}, \quad \psi_2(x) = \frac{x}{2} + \left(\frac{1}{2}, 0\right), \quad \psi_3(x) = \frac{x}{2} + \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right).$$

From Hutchinson's result (see Hutchinson, 1981) there exists a unique nonvoid compact set $\mathscr{G}^0 \subset \mathbb{R}^2$ such that

$$\mathscr{G}^0 = \psi_1(\mathscr{G}^0) \cup \psi_2(\mathscr{G}^0) \cup \psi_3(\mathscr{G}^0).$$

This set is called *the unit Sierpiński gasket*. Let $\mathscr{G}_0$ be its two-sided version, i.e.

$$\mathscr{G}_0 = \mathscr{G}^0 \cup \widetilde{\mathscr{G}^0},$$

where $\widetilde{\mathscr{G}^0}$ is the reflection of $\mathscr{G}^0$ in the y-axis.

We need an unbounded version of $\mathscr{G}_0$. For $n = 0, 1, 2, \ldots$ let

$$\mathscr{G}_n = 2^n \mathscr{G}_0, \tag{1.1}$$

then the *unbounded Sierpiński gasket* is, by definition, the set

$$\mathscr{G} = \bigcup_{n=0}^{\infty} \mathscr{G}_n.$$

1.1. Notation related to the structure of $\mathcal{G}$

The gasket can be endowed with the natural shortest path metric, which in this case is equivalent to the Euclidean metric inherited from the plane. For $x, y \in \mathcal{G}$ their shortest path distance will be denoted by $d(x, y)$ and their Euclidean distance by $|x - y|$.

The Hausdorff dimension of the Sierpiński gasket is equal to

$$d_{\mathrm{f}} = \frac{\log 3}{\log 2}.$$

By μ we denote the d_{f}-dimensional Hausdorff measure on $\mathcal{G}^0$, normalized to have $\mu(\mathcal{G}^0) = 1$, as well as its natural extention to $\mathcal{G}$. This measure μ is a d_{f}-measure on the gasket, i.e. there exist two constants $0 < c_{1.1} \leqslant c_{1.2}$ such that

$$\forall_{r \in (0,1]} \ \forall_{z \in \mathcal{G}} \quad c_{1.1}\, r^{d_{\mathrm{f}}} \leqslant \mu(B(z,r)) \leqslant c_{1.2}\, r^{d_{\mathrm{f}}},$$

where $B(z,r)$ denotes the open ball in the metric d, with centre z and radius r.

Next, $d_{\mathrm{w}} = \log 5/\log 2$ is the so-called 'walk dimension' of the gasket, and

$$d_{\mathrm{s}} = \frac{2d_{\mathrm{f}}}{d_{\mathrm{w}}} = \frac{2\log 3}{\log 5}$$

its spectral dimension. Let us mention that $d_{\mathrm{s}} < 2$, which is decisive for the recurrence properties of the Brownian motion.

The infinite gasket $\mathcal{G}$ can be seen as a union of an infinite number of unit fractal triangles that are translates of $\mathcal{G}^0$, and the same holds true for $2^n\mathcal{G}^0$, $n = 1, 2, \ldots$. The collection of all 'gasket-triangles' with sidelength 2^n (translates of $2^n\mathcal{G}^0$) will be denoted by $\mathcal{T}_n$. These triangles meet only at their vertices and constitute natural building blocks of the infinite gasket. Any $\Delta \in \mathcal{T}_n$ will be called '*an n-triangle*'.

Finally, let V_0^∞ be the collection of all the vertices of 0-triangles. The set V_0^∞, called '*the* 0-*grid*', is a grid of points on the gasket, lying at distance 1 from each other.

1.2. Brownian motion on $\mathcal{G}$

Barlow and Perkins (1988) gave a construction of the process X_t, called the Brownian motion on the Sierpiński gasket (the construction of the Brownian motion on the Sierpiński gasket was earlier carried out by Goldstein (1987) and Kusuoka (1987)). This paper also furnishes very precise estimates for this process. It is a strong Markov and Feller process with continuous trajectories, which has a continuous symmetric density $p(t,x,y)$ with respect to the Hausdorff measure μ, satisfying (Theorem 1.5 of Barlow and Perkins, 1988): for each $t > 0$, $x, y \in \mathcal{G}$

$$\begin{aligned} &c_{1.3} t^{-d_{\mathrm{s}}/2} \exp\{-c_{1.4}(d(x,y)t^{-1/d_{\mathrm{w}}})^{d_{\mathrm{w}}/d_{\mathrm{w}}-1}\} \\ &\quad \leqslant p(t,x,y) \leqslant c_{1.5} t^{-d_{\mathrm{s}}/2} \exp\{-c_{1.6}(d(x,y)t^{-1/d_{\mathrm{w}}})^{d_{\mathrm{w}}/(d_{\mathrm{w}}-1)}\}. \end{aligned} \tag{1.2}$$

We have an estimate for the supremum of this process as well (Theorem 4.3 of Barlow and Perkins, 1988): for each $z \in \mathcal{G}$ and $t, \delta > 0$

$$P_z\left[\sup_{s \leqslant t} d(X_s, X_0) \geqslant \delta\right] \leqslant c_{1.7} \mathrm{e}^{-c_{1.8}(\delta t^{-1/d_{\mathrm{w}}})^{d_{\mathrm{w}}/(d_{\mathrm{w}}-1)}}.$$

The process admits a discrete scaling: for a Borel subset Γ of $\mathscr{G}$, for each $t>0$ and $x\in\mathscr{G}$

$$P_x[X_t\in\Gamma]=P_{2x}[\tfrac{1}{2}X_{5t}\in\Gamma].$$

The scaling in terms of its transition density reads (Theorem 7.8 of Barlow and Perkins, 1988):

$$\forall_{x,y\in\mathscr{G}}\ \forall_{t>0}\quad p(t,2x,2y)=\tfrac{1}{3}p(t/5,x,y). \tag{1.3}$$

Throughout the paper, $\bar{U}$ denotes the topological closure of the set U and *binary number* means an integer power of 2. Technical constants are labelled within each section by $c_{i.j}$.

2. The lower bound

At the beginning of this section we introduce the microscopic and macroscopic scales which will be appropriate for our setting.

Suppose that a binary number $R_0=2^{r_0}$ is chosen in such a way that

$$R_0^{d_\mathrm{f}}<\frac{d_\mathrm{f}}{v}.$$

R_0 is considered fixed throughout this section.

Next, let a_n be a binary number defined through

$$\tfrac{1}{2}(\log 2^n)^{1/d_\mathrm{f}}<a_n\leqslant(\log 2^n)^{1/d_\mathrm{f}} \tag{2.1}$$

and let r_n be an integer such that $R_0a_n=2^{r_n}$. Then we set the microscopic scale m_n and the macroscopic scale ℓ_n to be equal to

$$m_n=R_0a_n=2^{r_n},\quad \ell_n=2^n(R_0a_n)=2^{n+r_n}. \tag{2.2}$$

We have the following lemma.

Lemma 2.1. *$\boldsymbol{P}$-almost surely there exists $n_0=n_0(\omega)$ such that for $n\geqslant n_0(\omega)$ there exists an r_n-triangle $\Delta\subset\mathscr{G}_{n+r_n}$ which did not receive any Poisson points ($\mathscr{G}_k$ was defined by* (1.1)).

Proof. It is a simple consequence of the Borel–Cantelli lemma. We omit it. □

The next lemma estimates the supremum of the random potential V defined by (0.1) over $\mathscr{G}_{n+r_n}$.

Lemma 2.2. *$\boldsymbol{P}$-almost surely we have*

$$\sup_{x\in\mathscr{G}_{n+r_n}}V(x,\omega)=\mathrm{o}(\log 2^n),$$

as $n\to\infty$.

Proof. Again, it is an application of the Borel–Cantelli lemma. We prove that for arbitrary $\eta > 0$

$$\sum_n \boldsymbol{P}\left[\sup_{x\in\mathscr{G}_{n+r_n}} V(x,\omega) \geqslant \eta \log 2^n\right] < +\infty.$$

To establish this, observe that if for some 0-triangle Δ and integer k we have $\sup_{x\in\Delta} V(x,\omega) \geqslant k\|W\|_\infty$, then there must have been at least k Poisson points in the a-neighbourhood of Δ, therefore

$$\begin{aligned}\boldsymbol{P}\left[\sup_{x\in\mathscr{G}_{n+r_n}} V(x,\omega) \geqslant k\|W\|_\infty\right] &\leqslant \sum_{\mathscr{G}_{n+r_n}\supset\Delta\in\mathscr{T}_0} \boldsymbol{P}\left[\sup_{x\in\Delta} V(x,\omega) \geqslant k\|W\|_\infty\right] \\ &\leqslant \sum_{\mathscr{G}_{n+r_n}\supset\Delta\in\mathscr{T}_0} \boldsymbol{P}[\mathscr{N}(\Delta^a) \geqslant k]. \end{aligned} \tag{2.3}$$

Using $k = [\eta\log 2^n/\|W\|_\infty] + 1$ we see that (2.3) is a general term of a summable series and the lemma follows. □

The main result of this section is the following:

Theorem 2.3. *There exist positive constants C_1, C_2 such that $\boldsymbol{P}$-almost surely and for each $z \in \mathscr{G}$ and $t \geqslant 0$*

$$u^\omega(t,z) = E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right] \geqslant C_1 \exp\left\{-C_2\left(\frac{t}{(\log t)^{2/d_\mathrm{s}}} + \frac{t}{(\log t)^{2/d_\mathrm{s}+1/d_\mathrm{f}}}\right)\right\}.$$

Proof. Fix $z\in\mathscr{G}$ and choose ω outside the exceptional set from Lemma 2.1. If $n \geqslant n_0(\omega)$, then there exists an r_n-triangle free of Poisson points, located not too far from the origin, we consider this set to be a 'safe haven' for our process. To get the correct asymptotic lower bound we force the process to go to this set before time t and stay there without coming too close to its boundary, until t. This way there will be no interaction with the potential after the process enters deep enough into this triangle.

At this point we need to declare how n depends on t: for the given t let $n = n(t)$ be the unique integer satisfying

$$2^{n-1}(\log 2^{n-1})^p < t \leqslant 2^n(\log 2^n)^p, \tag{2.4}$$

where p is some number bigger than $1/d_\mathrm{f}$, to be determined later. Observe that if t is large enough, then $\ell_n < t$.

Fix $\Delta\subset\mathscr{G}_{n+r_n}$ to be an r_n-triangle free of Poisson points, assuming that t is large enough. Let $x_n \in \Delta$ be the midpoint of the left edge of Δ; $x_n \in V_0^\infty$ as long as $n \geqslant 1$. As we are interested in long-time asymptotics we can assume that the starting point z belongs to the V_0^∞, and if n is large enough, then $z \in \mathscr{G}_{n+r_n}$. The reason for this is that starting from z, we will hit some point from V_0^∞ after an asymptotically negligible time. It follows

$$d(x_n, z) \leqslant \operatorname{diam}\mathscr{G}_{n+r_n} = 2\cdot 2^{n+r_n} = 2\ell_n.$$

Let $\boldsymbol{v}_n$ be a gasket path joining x_n and z realizing their gasket distance. Since both $x_n, z \in V_0^\infty$ then $\boldsymbol{v}_n$ consists of $k_n \stackrel{\text{def}}{=} d(x_n,z)$ unit intervals joining points from the

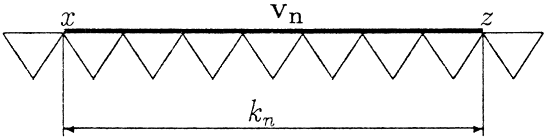

Fig. 1. The set V_n.

0-grid and moreover no two sides of any 0-triangle can belong to this path. Therefore, the set

$$V_n = \{x \in \mathscr{G}\colon d(x, \mathbf{v}_n) \leqslant 1\}$$

consists of $k_n + 2$ unit gasket triangles and is homeomorphic to the set sketched in Fig. 1.

We now make the process rush from z to x_n before time $\ell_n < t$, neither exiting the set V_n nor wandering too much within this set. Then we do not allow it to exit the ball $B(x_n, m_n/2 - a)$ until time t (a is the range of the potential).

Brownian motion on $\mathscr{G}$ is not translation invariant and we cannot apply any usual Girsanov-type transformation here. To bypass this difficulty let us impose the following conditions on our process:

1. $T_{\{x_n\}} \leqslant \ell_n$ ($T_{\{x_n\}}$ is the hitting time of $\{x_n\}$). As the process in point-recurrent, the $T_{\{x_n\}}$'s are finite a.s.
2. $T_{\{x_n\}} = T^{k_n}$, where $T^1, T^2, \dots$ are the consecutive hitting times of the 0-grid,

   $$T^1 = \inf\{t \geqslant 0\colon X_t \in \mathscr{G}_0 \backslash \{X_0\}\},$$

   $$T^{i+1} = T^i + \theta_{T^i} \circ T^1, \quad i = 1, 2, \dots .$$

   This condition means that the passage $z \mapsto x_n$ is accomplished in the smallest possible numbers of steps along the 0-grid.

Denote by $\mathscr{A}_n$ the event described in points 1 and 2 and by $\mathscr{B}_n$ the event

$$\mathscr{B}_n = \left\{ \sup_{T_{\{x_n\}} \leqslant s \leqslant t} d(X_s, x_n) < \frac{m_n}{2} - a \right\}.$$

As $\ell_n < t$, it follows that

$$E_z \left[\mathrm{e}^{-\int_0^t V(X_s, \omega)\,\mathrm{d}s} \right] \geqslant E_z \left[\mathrm{e}^{-\int_0^{\ell_n} V(X_s, \omega)\,\mathrm{d}s} 1_{\mathscr{A}_n} 1_{\mathscr{B}_n} \right]$$

and further, using the strong Markov property

$$\geqslant \mathrm{e}^{-\ell_n \sup_{x \in \mathscr{G}_{n+r_n}} V(X_s, \omega)} P_z[\mathscr{A}_n] P_{x_n} \left[\sup_{s \leqslant t} d(X_s, x_n) < \frac{m_n}{2} - a \right]. \tag{2.5}$$

From symmetry we have

$$P_z[\mathscr{A}_n] = \left(\frac{1}{4}\right)^{k_n} P_z[T^{k_n} < \ell_n].$$

To estimate the probability $P_z[T^k < s]$ we employ the reasoning that in Barlow and Perkins (1988) led to formula 4.1 (p. 577 of Barlow and Perkins, 1988).

We have the following Tauberian-type theorem (see Barlow and Perkins, 1988 or Fristedt and Pruitt, 1971).

If Y is a nonnegative random variable, then for each $\lambda > 0$, $t > 0$

$$P(Y \leqslant t) \geqslant \frac{E[\mathrm{e}^{-\lambda Y}] - \mathrm{e}^{-\lambda t}}{1 - \mathrm{e}^{-\lambda t}}. \tag{2.6}$$

This can be checked elementarily, too.

Using (2.6) with $t = \ell_n$ and applying the strong Markov property we obtain

$$P_z[\mathscr{A}_n] \geqslant \left(\frac{1}{4}\right)^{k_n} \frac{(E_0[\mathrm{e}^{-\lambda T^1}])^{k_n} - \mathrm{e}^{-\lambda \ell_n}}{1 - \mathrm{e}^{-\lambda \ell_n}}. \tag{2.7}$$

Recalling that (see 2.58 and 3.6 of Barlow and Perkins, 1988)

$$E_0[\mathrm{e}^{-\lambda T^1}] \geqslant \mathrm{e}^{-c_{2.1} \cdot 5 \cdot \lambda^{1/d_{\mathrm{w}}}}$$

and substituting

$$\lambda = (10 c_{2.1} k_n \ell_n^{-1})^{d_{\mathrm{w}}/(d_{\mathrm{w}}-1)}$$

in (2.7) we obtain

$$P_z[\mathscr{A}_n] \geqslant \frac{1}{2}\left(\frac{1}{4}\right)^{k_n} \exp\left\{-\frac{1}{2}(10 c_{2.1} k_n \ell_n^{-1})^{d_{\mathrm{w}}/(d_{\mathrm{w}}-1)} \ell_n\right\},$$

and, since $k_n = d(z, x_n) \leqslant 2\ell_n$,

$$P_z[\mathscr{A}_n] \geqslant c_{2.2} \mathrm{e}^{-c_{2.3}\ell_n}. \tag{2.8}$$

To estimate

$$P_{x_n}\left[\sup_{s \leqslant t} d(X_s, x_n) < \frac{m_n}{2} - a\right] = P_{x_n}[T_{B(x_n,(m_n/2)-a)} > t]$$

scale down this probability ($T_{B(x,r)}$ denotes the hitting time of the boundary of the ball $B(x,r)$) with an admissible number close to $(m_n/2 - a)$. Factor '5' comes from the fact that scaling is not exact. We obtain

$$P_{x_n}[T_{B(x_n,m_n/2-a)} > t] \geqslant \inf_{x \in \mathscr{G}} P_x\left[T_{B(x,1)} > \frac{5t}{(m_n/2 - a)^{d_{\mathrm{w}}}}\right].$$

For U — an open subset of $\mathscr{G}$, denote by $\lambda(U)$ the principal eigenvalue of the Brownian motion on U, killed upon coming to ∂U. T_U will denote the exit time from U. Asymptotically as $t \to \infty$, $P_x[T_{B(x,1)} > t]$ behaves as $c_{2.4}\mathrm{e}^{-\lambda(B(x,1))t}$ (see, for example, Pietruska-Pałuba, 1991, p. 14). As $t/m_n^{d_{\mathrm{w}}} \to \infty$ when $t \to \infty$, we have that, asymptotically, for each $x \in \mathscr{G}$

$$P_x\left[T_{B(x,1)} > \frac{5t}{(m_n/2 - a)^{d_{\mathrm{w}}}}\right] \geqslant c_{2.4} \exp\left\{-\lambda(B(x,1)) \frac{5t}{(m_n/2 - a)^{d_{\mathrm{w}}}}\right\}.$$

As $\sup_{x \in \mathscr{G}} \lambda(B(x,1)) < +\infty$ (an easy check), we obtain

$$P_{x_n}\left[\sup_{s \leqslant t} d(X_s, x_n) < \frac{m_n}{2} - a\right] \geqslant c_{2.4} \mathrm{e}^{-c_{2.5} t/m_n^{d_{\mathrm{w}}}}. \tag{2.9}$$

Collecting now (2.5), (2.8) and (2.9) we get that

$$E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right] \geqslant c_{2.6}\exp\left\{-\ell_n \sup_{x\in\mathscr{G}_{n+r_n}} V(x,\omega) - c_{2.3}\ell_n - c_{2.5}\frac{t}{m_n^{d_\mathrm{w}}}\right\}.$$

Lemma 2.2 provided an estimate of the supremum of the potential over $\mathscr{G}_{n+r_n}$, so that $\boldsymbol{P}$-almost surely, for large n (resp. for large t)

$$E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right] \geqslant c_{2.6}\exp\left\{-\ell_n[\mathrm{o}(\log 2^n) + c_{2.3}] - c_{2.5}\frac{t}{m_n^{d_\mathrm{w}}}\right\}. \tag{2.10}$$

Now choose

$$p = \frac{d_\mathrm{f} + d_\mathrm{w} + 3}{d_\mathrm{f}}$$

in the definition of $n(t)$, (2.4). This choice ensures that the leading term of this exponent will be $t/m_n^{d_\mathrm{w}}$.

To complete the proof, first investigate the difference

$$0 < \frac{t}{(m_{n(t)})^{d_\mathrm{w}}} - \frac{t}{R_0^{d_\mathrm{w}}(\log t)^{2/d_\mathrm{s}}},$$

approximately equal to

$$\frac{1}{R_0^{d_\mathrm{w}}}\frac{t}{(\log t)^{2/d_\mathrm{s}}}\left[\frac{(\log t)^{2/d_\mathrm{s}}}{(\log 2^{n(t)})^{2/d_\mathrm{s}}} - 1\right]. \tag{2.11}$$

Taking into account the definition of $n(t)$ (Eq. (2.4)), for $\bar{p} > p$ and for large t one has $\log 2^{n(t)} > \log t - \bar{p}\log\log t$. Hence (2.11) is smaller than

$$\frac{1}{R_0^{d_\mathrm{w}}}\frac{t}{(\log t)^{2/d_\mathrm{s}}}\left[\frac{1}{[1-(\bar{p}\log\log t/\log t)]^{2/d_\mathrm{s}}} - 1\right],$$

which behaves asymptotically when $t\to\infty$, up to a multiplicative constant, as

$$\frac{t}{(\log t)^{2/d_\mathrm{s}}}\frac{\log\log t}{\log t}. \tag{2.12}$$

We will be done if we show the following:

$$\limsup_{t\to\infty}\left(\ell_{n(t)}\log 2^{n(t)}\frac{(\log t)^{(1/d_\mathrm{f})+(2/d_\mathrm{s})}}{t}\right) < +\infty \tag{2.13}$$

and

$$\limsup_{t\to\infty}\frac{(\log t)^{(1/d_\mathrm{f})+(2/d_\mathrm{s})}}{t}\left(\frac{t}{(\log 2^n)^{2/d_\mathrm{s}}} - \frac{t}{(\log t)^{2/d_\mathrm{s}}}\right) < +\infty \tag{2.14}$$

(both limits will be zero, actually). Now (2.14) follows immediately from the asymptotics above, (2.12). To obtain (2.13) substitute the approximate value for ℓ_n, given by (2.2), getting an expression that converges to zero as long as $p > (d_\mathrm{w}+d_\mathrm{f}+2)/d_\mathrm{f}$. The proof is complete. □

3. The upper bound

In this section we will prove an upper bound on $u^{\omega}(t,z)=E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right]$, complementary to the one in Theorem 2.3. We will use the enlargement of obstacles method of Sznitman, adapted to the gasket case. We prove the following:

Theorem 3.1. *There exists a universal constant D such that **P**-almost surely and for each $z \in \mathscr{G}$*

$$\limsup_{t\to\infty} \frac{(\log t)^{2/d_s}}{t} E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right] < -D, \tag{3.1}$$

where $D>0$ is a universal constant which does not depend neither on z nor on the choice of the configuration ω outside the exceptional set.

The proof of this theorem proceeds as follows. First, we enlarge the obstacles and remove enlarged obstacles from the state space. We are able to control the principal eigenvalue of the 'free open set', at least as long as it is neither too large nor too small (Theorem 3.2). To make the theorem work for us we need to link the integrals involving the potential with the estimates of eigenvalues. This is done by introducing independent Poisson processes (i.e. 'clocks' ticking at exponential times) at obstacle sites and by relating these processes to our integral. See the discussion just before Theorem 3.2.

The volume of the 'free open set' resulting from the enlargement of obstacles can be controlled, which allows us to control the principal eigenvalues as well (Proposition 3.6). Proof of Theorem 3.1 is then straightforward in this context.

To start with, let us explain the enlargement of obstacles method. Similarly to the lower bound proof, the unit (intermediate) length scale a_n will be roughly equal to $(\log 2^n)^{1/d_f}$ — precisely, a_n will be a binary number defined by (2.1). Let

$$a_n = 2^{\rho_n}. \tag{3.2}$$

First, for fixed $b \gg a$, (a is the range of the potential funcion) remove from the gasket those closed triangles of size b that have received some of the obstacle points. Next, we take a look at the bigger gasket triangles — of size a_n. If we removed a large portion of any of those triangles, we remove this triangle as a whole. Otherwise, we keep it. And finally, from those triangles that we kept, we remove the obstacles, i.e. balls with radius b, centred at Poisson points.

Formally, let $r>0$ be fixed. Suppose $\Delta \in \mathscr{T}_{\rho_n}$ is a fixed ρ_n-triangle. We chop the sides of this triangle into segments of length b each, which yields $(a_n/b)^{d_f}$ smaller gasket triangles (of size b each).

Let $U(\Delta)$ be the open subset of Δ obtained by removing those closed small triangles where some of the Poisson points fell. Then one says that Δ *is of clearing type* (or, that there is a clearing of size r inside Δ) if the resulting set $U(\Delta)$ is large enough, compared to the volume of Δ; if

$$\mu(U(\Delta)) > \frac{\omega(r)}{2}\mu(\Delta),$$

where $\omega(r)=\inf_{x\in\mathcal{G}}\mu(B(x,r))\geqslant c_{1.1}r^{d_{\mathrm{f}}}$. Consequently, we write

$$\mathrm{Cl}(\Delta)=\left\{\omega\in\Omega:\ \mu(U(\Delta))>\frac{\omega(r)}{2}\mu(\Delta)\right\}$$

($\mathrm{Cl}(\Delta)$ consists of those configurations ω which produce a clearing inside Δ).

If Δ is a given ρ_n-triangle, then the probability of $\mathrm{Cl}(\Delta)$ can be bounded above as follows:

$$\begin{aligned}\boldsymbol{P}[\mathrm{Cl}(\Delta)]&=\boldsymbol{P}\left[\mu(U(\Delta))\geqslant\frac{\omega(r)}{2}\mu(\Delta)\right]\\&\leqslant\#\{\text{configurations of small triangles}\}\\&\qquad\times\boldsymbol{P}\left[\text{no Poisson point inside the given set of measure }\frac{\omega(r)}{2}\mu(\Delta)\right]\\&=2^{(a_n/b)^{d_{\mathrm{f}}}}\mathrm{e}^{-\nu(\omega(r)/2)a_n^{d_{\mathrm{f}}}}.\end{aligned}\tag{3.3}$$

Let $B^{(n)}=B^{(n)}(\omega)$ be the clearing set at scale a_n: union of those closed ρ_n-triangles where clearing is present, i.e.

$$B^{(n)}=\bigcup\{\bar{\Delta}:\ \Delta\text{ is of clearing type}\}$$

and let $\widetilde{B^{(n)}}$ be the open subset of $\mathcal{G}$ consisting of those points that are closer than one length unit from $B^{(n)}$, i.e.

$$\widetilde{B^{(n)}}=\{x\in\mathcal{G}:\ d(x,B^{(n)})<a_n\}.$$

Let $\mathcal{P}$ be an open subset of $\mathcal{G}$. *The free open set* $\mathcal{O}_{b,\mathcal{P}}^{(n)}$ inside $\mathcal{P}$ is defined as

$$\mathcal{O}_{b,\mathcal{P}}^{(n)}=(\mathcal{P}\cap\widetilde{B^{(n)}})\Big\backslash\bigcup_{x_i\in\mathcal{N}}\bar{B}(x_i,b).\tag{3.4}$$

Recall that for an open subset $\mathcal{P}\subset\mathcal{G}$, $T_{\mathcal{P}}$ stands for the entrance time into $\mathcal{P}^{\mathrm{c}}$.

We need to relate integrals involving the potential to some principal eigenvalue estimates. To this end let us introduce an exponential moment for $T\wedge T_{\mathcal{P}}$, where T will be the death moment of the process X_t, as introduced below. We will be in a situation much alike the one with killing Poisson obstacles, at least as far as the estimates are concerned.

Let us attach to each point of the cloud an exponential clock and then kill the process once the quantity $A_t^i\stackrel{\mathrm{def}}{=}\int_0^t W(X_{\mathrm{s}}-x_i)\,\mathrm{d}s$ becomes bigger than this clock. Formally, to each of the points x_i we connect a canonical cadlag process $(N_t^i)_{t\geqslant 0}$ in such a way that they are all independent and independent of the Brownian motion. For precise definition (see Sznitman, 1993b).

Then

$$T_i\stackrel{\mathrm{def}}{=}\inf\{s\geqslant 0:\ N_t^i(A_s^i)\geqslant 1\}$$

is the death time of the process associated with the ith point of the cloud, x_i. The actual death time of the process X_t is then defined as

$$T=\inf_i T_i,$$

and

$$P_z[T > t|X.] = P_z[\forall i T_i > t|X.]$$
$$= \prod_i P_z[T_i > t|X.] = \prod_i \mathrm{e}^{-\int_0^t W(X_s - x_i)\,\mathrm{d}s} = \mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}. \tag{3.5}$$

Next, we scale the picture and rewrite the process in new space/time units — a_n will be the new space unit and consequently $a_n^{d_w}$ — the new time unit. The rescaled stopping times will be denoted by $T_i^{(n)}$, $T^{(n)}$ and $T_{\mathscr{P}}^{(n)}$ and the rescaled free open set by $\widetilde{\mathcal{O}_{b,\mathscr{P}}^{(n)}}$. This new free open set corresponds to obstacles with radius b/a_n and intensity $va_n^{d_f}$, and to 1-neighbourhood rather than a_n-neighbourhood of the clearing set at scale 1, inside the open set $1/a_n\,.\mathscr{P}$. $T_{\mathscr{P}}^{(n)}$ is the exit time from the set $1/a_n\,.\mathscr{P}$. We should also change the function W to $W_n(\cdot) \stackrel{\text{def}}{=} a_n^{d_w} W(a_n\cdot)$ and the configuration, starting point and the set $\mathscr{P}$ to the rescaled ones, but this change can be omitted since our result will be uniform in z, ω, W and $\mathscr{P}$.

We now formulate the gasket counterpart of Theorem 2.1 from Sznitman (1993b). Its proof is almost identical to the proof in Euclidean-space, so we omit it.

Theorem 3.2. *Let $M > 0$ and $\delta > 0$ be given. Then, in the setting as above,*

$$\limsup_{r\to 0} \sup_{b>a,\, b \text{ binary}} \sup_W \limsup_{n\to\infty} \sup_{z,\omega,\mathscr{P}} E_z\left[\mathrm{e}^{(\lambda(\widetilde{\mathcal{O}_{b,\mathscr{P}}^{(n)}})\wedge M-\rho)(T_{\mathscr{P}}^{(n)}\wedge T^{(n)})}\right] \leqslant D(M,\rho)$$

for some constant $D(M,\rho) > 0$.

Proof. Omitted.

Observe that scaling back we obtain

$$E_z\left[\mathrm{e}^{(\lambda(\widetilde{\mathcal{O}_{b,\mathscr{P}}^{(n)}})\wedge M-\rho)(T^{(n)}\wedge T_{\mathscr{P}}^{(n)})}\right] = E_z\left[\mathrm{e}^{\bar{\lambda}(T\wedge T_{\mathscr{P}})}\right],$$

where $\bar{\lambda} = \lambda(\mathcal{O}_{b,\mathscr{P}}^{(n)}) \wedge M/a_n^{d_w} - \rho/a_n^{d_w}$. Careful integration by parts shows that this last quantity is equal to

$$E_z\left[\left\{\int_0^{T_{\mathscr{P}}} V(X_s,\omega)\mathrm{e}^{-\int_0^s V(X_u,s)\,\mathrm{d}u+\bar{\lambda}s}\,\mathrm{d}s\right\} + \mathrm{e}^{-\int_0^{T_{\mathscr{P}}} V(X_s,\omega)\,\mathrm{d}s+\bar{\lambda}T_{\mathscr{P}}}\right].$$

Therefore as a corollary we get, for our potential V:

Corollary 3.3. *Let $M > 0$, $\delta > 0$ be given. Then*

$$\limsup_{r\to 0} \sup_{b>a,\, b \text{ binary}} \limsup_{n\to\infty} \sup_{z,\omega,\mathscr{P}}$$
$$E_z\left[\left\{\int_0^{T_{\mathscr{P}}} V(X_s,\omega)\mathrm{e}^{-\int_0^s V(X_u,s)\,\mathrm{d}u+\bar{\lambda}s}\,\mathrm{d}s\right\} + \mathrm{e}^{-\int_0^{T_{\mathscr{P}}} V(X_s,\omega)\,\mathrm{d}s+\bar{\lambda}T_{\mathscr{P}}}\right]$$
$$\leqslant D(M,\rho) < \infty, \tag{3.6}$$

where $\bar{\lambda}=\lambda(\mathcal{O}_{b,\mathcal{P}}^{(n)})\wedge M/a_n^{d_w}-\rho/a_n^{d_w}$, $T_{\mathcal{P}}$ *denotes the entrance time into* $\mathcal{P}^c$ *and* $D(M,\rho)$ *is the constant from Theorem* 3.2.

Estimate (3.6) is uniform in $\mathcal{P}$. Therefore it holds in particular for

$$\mathcal{P}_n = \operatorname{Int}\mathcal{G}_{n+\rho_n}.$$

Till the end of this paper we will use these sets $\mathcal{P}_n$ instead of the general $\mathcal{P}$, so that we may drop $\mathcal{P}_n$ from the notation — we write $\mathcal{O}_b^{(n)}$ instead of $\mathcal{O}_{b,\mathcal{P}_n}^{(n)}$.

Now we turn to the lower bound estimate for the principal eigenvalue of the free open set $\mathcal{O}_b^{(n)}$. First, we need two lemmas about the geometrical structure of this set.

Lemma 3.4. *Let* $b>0$ *and* $r>0$ *be the given numbers that satisfy*

$$\frac{v\omega(r)}{2^{1+d_f}}-\frac{\log 2}{b^{d_f}}>0. \tag{3.7}$$

Then there exist $\gamma=\gamma(b,r)>0$ and $m_0=m_0(b,r)>0$ *such that the following statement holds true*:

***P**-almost surely there exists* $n_0(\omega)$ *such that for* $n\geqslant n_0(\omega)$ *the number of clearing* ρ_n*-triangles included within distance* $2^{n\gamma}$ *from the given one does not exceed* m_0 (‘*distance*’ *is understood as Hausdorff distance between the closed sets*).

Proof. If $A\geqslant 0$ is given, then a simple volume comparison shows that there is at most $c_{3.1}(A/a_n)^{d_f}$ ρ_n-triangles within distance A from a given ρ_n-triangle. Therefore for any integer $m_0>0$ the probability of finding more that m_0 clearing ρ_n-triangles within distance A from the given one does not exceed (use (3.3))

$$\left(c_{3.1}\left(\frac{A}{a_n}\right)^{d_f}\right)^{m_0}\left(2^{(a_n/b)^{d_f}}\mathrm{e}^{-(v\omega(r)/2)a_n^{d_f}}\right)^{m_0}.$$

Substituting $A=2^{n\gamma}$ with $\gamma=1/2d_f((v\omega(r)/2)-\log 2/b^{d_f})$ and $m_0=[1/\gamma]+1$ we obtain a term of a convergent series. The statement follows now from the Borel–Cantelli lemma. □

Having proven this lemma we can estimate the size of the connected component of $\widetilde{B^{(n)}}$ containing the given clearing ρ_n-triangle.

Lemma 3.5. *Let* r *and* b *satisfy* (3.7). *Then* ***P**-almost surely there exists* $n_1(\omega)\geqslant n_0(\omega)$ *such that for* $n\geqslant n_1(\omega)$ *the connected component of* $\widetilde{B^{(n)}}$ *containing the given* ρ_n*-triangle* $\Delta\subset\mathcal{P}_n$ *of clearing type is included in*

$$V(\Delta)=\operatorname{Int}\left(\bigcup_{C\in\mathcal{T}_{\rho_n},\,\rho_H(C,\Delta)\leqslant 2m_0a_n}\bar{C}\right)$$

(ρ_H *denotes the Hausdorff distance between compact subsets of* $\mathcal{G}$).

Proof. Let $n_1(\omega)\geqslant n_0(\omega)$ be large that $2^{n_1\gamma}>2m_0$ and let $n>n_1$. We know that within distance $A=2^{n\gamma}$ there are at most m_0 clearing triangles, so that the connected set in question is built of at most m_0 triangular tiles of size 2^{ρ_n}. How big a connected set can be built from open a_n-neighbourhoods of such tiles? Those tiles must not lie at 'distance' larger than one unit from each other, for no gap of width $2a_n$, or more, can be filled. Therefore, the farthest triangle in this set can lie only as far as $2m_0a_n$ from the given one. The lemma is established. □

The two lemmas enable us to get a lower bound on $\lambda(\mathcal{O}_b^{(n)})$.

Proposition 3.6. *Let r and b satisfy* (3.7). *Then $\boldsymbol{P}$-almost surely there exists $n_2 = n_2(\omega)\geqslant n_1(\omega)$ such that for $n\geqslant n_2(\omega)$*

$$\lambda(\mathcal{O}_b^{(n)}(\omega))\geqslant c_{3.3}\left[\frac{2n\log 2}{v}\left(d_{\mathrm{f}}+c_{3.2}\left(\frac{m_0}{b}\right)^{d_{\mathrm{f}}}\right)\right]^{-2/d_{\mathrm{s}}},$$

where $c_{3.2}, c_{3.3} > 0$ are universal constants and m_0 is the constant from Lemma 3.4.

Proof. Assume that $n\geqslant n_1(\omega)$ and that ω does not belong to the exceptional sets from the lemmas above. Each connected component of the free open set

$$\mathcal{O}_b^{(n)}(\omega)=(\mathcal{P}_n\cap\widetilde{B^{(n)}})\Big\backslash\bigcup_{x_i\in\mathcal{N}}\bar{B}(x_i,b)$$

is included in a connected component of $\mathcal{P}_n\cap\widetilde{B^{(n)}}$ and therefore also in $V(\Delta)\backslash\bigcup_{x_i\in\mathcal{N}}\bar{B}(x_i,b)$ for some ρ_n-triangle Δ of clearing type. As

$$V(\Delta)=\mathrm{Int}\left[\bigcup_{C\in\mathcal{T}_{\rho_n},\rho_H(C,\Delta)\leqslant 2m_0a_n}\bar{C}\right],$$

we have

$$V(\Delta)\Big\backslash\bigcup_{x_i\in\mathcal{N}}\bar{B}(x_i,b)\subseteq V_1(\Delta);$$

$V_1(\Delta)$ stands for the complement in $V(\Delta)$ of those closed subtriangles of size b in some ρ_n-triangle C with $\rho_H(C,\Delta)\leqslant 2m_0a_n$ where some of the Poisson points fell. Again, there is at most $c_{3.1}(m_0)^{d_{\mathrm{f}}}$ ρ_n-triangles in $V(\Delta)$ so that for every Δ as above and for each $v\in\mathbb{R}_+$ one has

$$\begin{aligned}\boldsymbol{P}[\mu(V_1(\Delta))\geqslant v] &\leqslant \#\{\text{all possible choices of triangles}\}\\ &\qquad\times\#\left\{\begin{matrix}\text{all possible configurations}\\ \text{of triangles removed}\end{matrix}\right\}\\ &\qquad\times\boldsymbol{P}[\text{given set of measure } v \text{ did not receive any Poisson points}]\\ &\leqslant 2^{c_{3.1}(m_0)^{d_{\mathrm{f}}}}2^{(a_n/b)^{d_{\mathrm{f}}}}\mathrm{e}^{-\nu v}. \end{aligned}\tag{3.8}$$

Taking into account all possible choices of Δ and using (3.8) we get

$$\boldsymbol{P}[\exists\Delta\subset\mathcal{P}_n,\Delta\in\mathcal{T}_{\rho_n}:\mu(V_1(\Delta))\geqslant v]\leqslant 2^{nd_{\mathrm{f}}}2^{c_{3.1}(m_0)^{d_{\mathrm{f}}}(\frac{a_n}{b})^{d_{\mathrm{f}}}}\mathrm{e}^{-\nu v}$$

which for

$$v = v_n = \frac{2n\log 2}{\nu}\left(d_{\mathrm{f}} + c_{3.2}\left(\frac{m_0}{b}\right)^{d_{\mathrm{f}}}\right) \tag{3.9}$$

is a term of a convergent series. From the Borel–Cantelli lemma we infer that there exists $n_2(\omega) \geqslant n_1(\omega)$ such that for $n \geqslant n_2$ the measure of each connected component of $\mathcal{O}_b^{(n)}$ does not exceed v_n.

Let $\bar{v}_n$ now be the smallest binary number not smaller than $v_n^{1/d_{\mathrm{f}}}$:

$$\frac{\bar{v}_n}{2} < v_n^{1/d_{\mathrm{f}}} \leqslant \bar{v}_n.$$

Then the measure of each connected component of $\mathcal{O}_b^{(n)}$ does not exceed $\bar{v}_n^{d_{\mathrm{f}}}$ as well. As the transition density of the process scales according to (1.3), for the principal eigenvalues we will have:

$$\forall_{U \subset \mathcal{G}, U \text{ open}} \quad \lambda(2U) = \tfrac{1}{5}\lambda(U).$$

From this fact and from the monotonicity of principal eigenvalues we have

$$\lambda(\mathcal{O}_b^{(n)}) = \lambda\left(\frac{\mathcal{O}_b^{(n)}}{\bar{v}_n}\bar{v}_n\right)$$

$$\geqslant \frac{1}{\bar{v}_n^{d_{\mathrm{w}}}} \inf_{U \in \mathcal{M}_{\leqslant 1}} \lambda(U) \geqslant \frac{1}{5 v_n^{2/d_{\mathrm{s}}}} \inf_{U \in \mathcal{M}_1} \lambda(U),$$

where $\mathcal{M}_{\leqslant 1} = \{U \subset \mathcal{G}\colon U \text{ open}, \ \mu(U) \leqslant 1\}$ and $\mathcal{M}_1 = \{U \subset \mathcal{G}\colon U \text{ open}, \ \mu(U) = 1\}$. Setting $c_{3.3} = \frac{1}{5}\{\inf_{U \in \mathcal{M}_1} \lambda(U)\} > 0$ (the infimum is positive by the trace formula) we get

$$\lambda(\mathcal{O}_b^{(n)}) \geqslant \frac{c_{3.3}}{v_n^{2/d_{\mathrm{s}}}},$$

which is exactly the statement of the proposition. □

Finally, we are ready to obtain the upper bound.

Proof of Theorem 3.1. We use the estimate from Corollary 3.3 together with the lower bound for $\lambda(\mathcal{O}_b^{(n)})$ from Proposition 3.6. For given $M > 0$ and $\delta > 0$ the claim of Corollary 3.3 holds with

$$\bar{\lambda} = \lambda(\mathcal{O}_b^{(n)}) \wedge \frac{M}{a_n^{d_{\mathrm{w}}}} - \frac{\delta}{a_n^{d_{\mathrm{w}}}}$$

and since $\lambda(\mathcal{O}_b^{(n)}) \geqslant c_{3.3}/v^{2/d_{\mathrm{s}}}$ (v was defined by (3.9)) it will also hold, at least for large n and an appropriate choice of M, with

$$\lambda = \left(\frac{c_{3.3}}{v^{2/d_{\mathrm{s}}}} - \frac{\delta}{a_n^{d_{\mathrm{w}}}}\right)_+ = \frac{1}{a_n^{d_{\mathrm{w}}}}\left(c_{3.4}\left(\frac{\nu}{d_{\mathrm{f}} + c_{3.2}(m_0/b)^{d_{\mathrm{f}}}}\right)^{2/d_{\mathrm{s}}} - \delta\right)_+.$$

If we relate t to n via

$$2^{n(t)} \leqslant t < 2^{n(t)+1}, \tag{3.10}$$

then $\boldsymbol{P}$-almost surely

$$\limsup_{r\to 0}\ \limsup_{b\to\infty,\,\text{binary}}\ \limsup_{n(t)\to\infty}\ \sup_{z}$$

$$E_z\left[\int_0^{T_{\mathscr{P}_n}}\left\{V(X_s,\omega)\mathrm{e}^{-\int_0^s V(X_u,\omega)\,\mathrm{d}u+\lambda s}\,\mathrm{d}s\right\}+\mathrm{e}^{-\int_0^{T_{\mathscr{P}_n}} V(X_s,\omega)\,\mathrm{d}s+\lambda T_{\mathscr{P}_n}}\right]\leqslant D(M,\rho). \tag{3.11}$$

Set

$$\mathscr{A}_t\overset{\text{def}}{=}E_z\left[1\{T_{\mathscr{P}_n}>t\}\mathrm{e}^{-\int_0^t V(X_u,\omega)\,\mathrm{d}u+\lambda t}\right].$$

For each $t>0$, $\mathscr{A}_t$ is not larger than the quantity under the limit operations in (3.11). Moreover,

$$E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right]\leqslant P_z[T_{\mathscr{P}_n}\leqslant t]+A\mathrm{e}^{-\lambda t}. \tag{3.12}$$

It is clear that

$$P_z[T_{\mathscr{P}_n}\leqslant t]\leqslant P_z\left[\sup_{s\leqslant t} d(X_0,X_s)\geqslant d(z,\partial\mathscr{P}_n)\right]\leqslant c_{1.7}\mathrm{e}^{-c_{1.8}(d(z,\partial\mathscr{P}_n)t^{-1/d_\mathrm{w}})^{d_\mathrm{w}/(d_\mathrm{w}-1)}}.$$

If n is large enough then $d(z,\partial\mathscr{P}_n)\geqslant\frac{1}{2}2^{n+\rho_n}$, so that, in view of (3.10)

$$d(z,\partial\mathscr{P}_{n(t)})\geqslant\tfrac{1}{8}t(\log t)^{1/d_\mathrm{f}}.$$

Therefore,

$$\frac{(\log t)^{2/d_\mathrm{s}}}{t}\log P_z[T_{\mathscr{P}_{n(t)}}\leqslant t]\leqslant -c_{3.5}(\log t)^{(2/d_\mathrm{s})(1/(d_\mathrm{w}-1)-1)}\overset{t\to\infty}{\longrightarrow}-\infty$$

which allows us to write

$$\limsup_{t\to\infty}\frac{(\log t)^{2/d_\mathrm{s}}}{t}\log E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right]$$

$$\leqslant\text{‘lims’}\,\frac{(\log t)^{2/d_\mathrm{s}}}{t}\log(\mathscr{A}_t\mathrm{e}^{-\lambda t})$$

$$=\text{‘lims’}(-\lambda(\log t)^{2/d_\mathrm{s}})$$

(‘lims’ denotes the limit operations from (3.11)). As

$$\lambda(\log t)^{2/d_\mathrm{s}}\geqslant c_{3.6}\left(\left(\frac{v}{d_\mathrm{f}+c_{3.2}(m_0/b)^{d_\mathrm{f}}}\right)^{2/d_\mathrm{s}}-\delta\right)_+,$$

taking ‘lims’ we obtain

$$\limsup_{t\to\infty}\frac{(\log t)^{2/d_\mathrm{s}}}{t}\log E_z\left[\mathrm{e}^{-\int_0^t V(X_s,\omega)\,\mathrm{d}s}\right]\leqslant -c_{3.6}\left\{\left(\frac{v}{d_\mathrm{f}}\right)^{2/d_\mathrm{s}}-\delta\right\}_+$$

with some positive constant $c_{3.6}$. Choosing $\delta=\frac{1}{2}(v/d_\mathrm{f})^{2/d_\mathrm{s}}>0$ and $D=c_{3.6}\cdot\delta$ we conclude the proof of the theorem. □

Collecting now the results of Theorems 2.3 and 3.3 we get the statement of Theorem 0.1.

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